Algorithmic schemes for the multiple-sets split equality problem

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ABSTRACT

In this paper, for solving the multiple-sets split equality problem (MSSEP), we give a general approach to construct iterative methods. We present an weakly convergent string-averaging algorithmic scheme, that contain the cyclic and simultaneous iterative methods as particular cases. In our methods, we do not need to have any information on the operator norms. We also give numerical examples for illustrating two main methods.

Lược đồ thuật toán giải bài toán trùng tách đa tập

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THÔNG TIN BÀI BÁO

Ngày nhận bài:
Ngày hoàn thiện:
Ngày đăng:

TỪ KHÓA

Không gian Hilbert
Phép chiếu Metric
Không giãn

TÓM TẮT

Trong bài báo này, để giải bài toán trùng tách nhiều tập, chúng tôi đưa ra một cách tiếp cận tổng quát để xây dựng phương pháp lặp. Chúng tôi giới thiệu một lược đồ thuật toán xâu-trung bình hội tụ yếu, chứa phương pháp lặp xoay vòng và phương pháp đồng thời như các trường hợp riêng. Ở đây chúng tôi không yêu cầu biết chuẩn của toán tử. Chúng tôi cũng đưa ra ví dụ số minh họa cho hai phương pháp cơ bản.

1. Introduction

Let H_1, H_2 and H_3 be real Hilbert spaces. Let J_1 and J_2 be two index sets with N and M elements. Let $\{C_i\}_{i\in J_1}$ and $\{Q_j\}_{j\in J_2}$ be two families of closed convex subsets in H_1 and H_2, respectively, and let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear mappings with the standard norms $\|A\|$ and $\|B\|$, respectively. We denote by I, $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ the the identity mapping, an inner product and a norm for any Hilbert space.

The multiple-sets split equality problem is to find a point $z_* = [x_*, y_*]$ with the

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property:

$$x_* \in C := \cap_{i\in J_1} C_i \quad \text{and} \quad y_* \in Q := \cap_{j\in J_2} Q_j \quad \text{such that} \quad Ax_* = By_*. \tag{1.1}$$

Denote by Γ the set of solutions for (1.1), assumed to be non-empty in this paper.

Clearly, when $H_2 = H_3$ and $B = I$, the MSSEP reduces to the *multiple-sets split feasibility problem* (MSSFP), that was first introduced by Censor and Elfving [1] for modeling inverse problems that arise from phase retrievals and in image reconstruction [2]. Recently, the MSSFP can also be used to model the intensity-modulated radiation therapy [3,4] and references therein.

In the case that $N = M = 1$, the MSSEP reduces to the split equality problem (SEP), that is to find points x_* and y_* such that

$$x_* \in C, \quad y_* \in Q \quad \text{and} \quad Ax_* = By_*. \tag{1.2}$$

Problem (1.2) was introduced and studied by Byrne and Moudafi [7] in finite-dimensional spaces. This is actually an optimization problem with weak coupling in the constraint and its interest covers many situations, for instance, in domain decomposition for PDEs [8] and game theory [9]. In order to solve problem (1.2), they introduced the weakly convergent CQ-like method, $z^1 = [x^1, y^1] \in C \times Q$ and

$$\begin{aligned} x^{k+1} &= P_C(x^k - \gamma_k A^*(Ax^k - By^k)), \\ y^{k+1} &= P_Q(x^k + \gamma_k B^*(Ax^k - By^k)),\ \forall k \geq 1, \end{aligned} \tag{1.3}$$

where A^* and B^* are the adjonts of A and B, respectively, and $\gamma_k = \gamma$ is chosen in the interval $(a, b) \subset (0, \min\{1/\|A\|^2, 1/\|B\|^2\})$ for all $k \geq 1$. So, the choice value γ depends on the norms $\|A\|$ and $\|B\|$, that are not easy to be calculated in practice. To overcome the difficulty, Dong et al. [8] and Vuong et al. [9] indicated that γ_k can be chosen by

$$\gamma_k = \frac{\rho_k f(x^k, y^k)}{a_k} \quad \text{with } \rho_k \in (0, 4), \tag{1.4}$$

where $f(x, y) = \|Ax - By\|^2/2$ and $a_k = \|A^*(Ax^k - By^k)\|^2 + \|B^*(Ax^k - By^k)\|^2$. Next, Chuang and Du [10] proved weak convergence for method (1.3) when γ_k is chosen in the interval $(0, 2/(\|A\|^2 + \|B\|^2))$ such that $\liminf_{k\to\infty} \gamma_k(2 - \gamma_k(\|A\|^2 + \|B\|^2)) > 0$ with an additional conditions on (x^k, y^k). Recently, Wang [11] gave a new way to select the parameter γ_k. The iterative regularization method and several projection methods have been investigated in [12-15].

Clearly, in the Hilbert space $H = H_1 \times H_2$ with an inner product and a norm denoted and defined by $\langle z^1, z^2\rangle = \langle x^1, x^2\rangle + \langle y^1, y^2\rangle$ and $\|z\| = (\|x\|^2 + \|y\|^2)^{1/2}$, respectively, where $z = [x, y]$ and $z^i = \langle x^i, y^i\rangle$ with $x, x^i \in H_1$ and $y, y^i \in H_2$ for $i = 1, 2$, method (1.3) can be re-written in the compact form,

$$z^{k+1} = P_S(I - \gamma G^* G) z^k,\ z^1 \in H, \tag{1.5}$$

where $G = [A, -B]^T : H \to H$ and $S = C \times Q$. Further, Li and Chen [16] extended (1.5) to MSSEP (1.1) with $N > M$ by a sequential iterative method,

$$z^{k+1} = P_{S_{m(k)}}(I - \gamma G^* G) z^k, \tag{1.6}$$

where $m(k) = k \mod (N+1)$ with $Q_j = H_2$, for $M < j \leq N$, is some additional set and $S_i = C_i \times Q_i$ for $i = 1, \cdots, N$, and a simultaneous one. They proposed also several iterative methods of Krasnoselskii-Mann's type. All these methods converge weakly to a point in Γ. Further, Zhao and Shi [17] introduced a new extragradient-type method for the MSSEP. Meantime, Tian et al. [18] proposed a new iterative method, in which the iterative step size is split self-adaptive without needing to have any information about $\|A\|$ and $\|B\|$.

When $H_1 = H_2 = H_3$ and $A = B = I$, problem (1.1) reduces to the convex feasibility problem, that is to find a point $p_* \in \cap_{i=1}^{n} C_i$ where n is a positive integer and C_i is a closed convex set in a Hilbert space H for all $1 \leq i \leq n$. To solve the convex feasibility problem, Censor et al [19] introduced a *string-averaging* algorithmic scheme, that projects a point sequentially along several independent strings of constraints. Projecting along each string is sequential, but the strings are independent and projecting along them can be performed in parallel. In final, the end-points of strings of sequential projections onto the constraints are averaged.

The purpose of this paper is to use the results listed above to design a general scheme for iterative methods, solving (1.1). The rest of this paper is organized as follows. In Section 2, we list some related facts, that will be used in the proof of our results. In Section 3, we propose a string-averaging scheme to solve (1.1) and show its weak convergence. Finally, in Section 4, we give numerical experiments for illustrating our main results.

2. Preliminaries

In any real Hilbert space H, we have the following inequality,

$$\|u+v\|^2 \leq \|u\|^2 + 2\langle v, u+v\rangle, \ \forall u, v \in H.$$

Definitions 2.1 A mapping T from a subset Ω of H into H is called:
(i) nonexpansive, if $\|Tu - Tv\| \leq \|u - v\|$ for all $u, v \in \Omega$;
(ii) contractive, if $\|Tu - Tv\| \leq \tilde{a}\|u - v\|$ for a fixed $\tilde{a} \in [0, 1)$ and for all $u, v \in \Omega$;
(iii) γ-inverse strongly monotone, if $\gamma\|Tu - Tv\|^2 \leq \langle Tu - Tv, u - v\rangle$ for all $u, v \in \Omega$, where γ is a positive number;
(iv) firmly nonexpansive, if there holds (iii) with $\gamma = 1$.
(v) η-strongly monotone and γ-strictly pseudocontractive mapping, if there hold, respectively,

$$\begin{aligned}&\langle Tx_1 - Tx_2, x_1 - x_2\rangle \geq \eta\|x_1 - x_2\|^2 \quad \text{and}\\ &\langle Tx_1 - Tx_2, x_1 - x_2\rangle \leq \|x_1 - x_2\|^2 - \gamma\|(I - T)x_1 - (I - T)x_2\|^2\end{aligned}$$

for all $x_1, x_2 \in \Omega$, where η and γ are some positive real numbers.

For a closed convex subset Ω of H, there exists a mapping $P_\Omega : H$ onto Ω such that $P_\Omega(u) = \inf_{v\in\Omega} \|v-u\|$ for each $u \in H$. The mapping P_Ω is called the metric projection onto Ω. We know that P_Ω is firmly nonexpansive (hence, nonexpansive); $I - P_\Omega$ is also firmly nonexpansive; $\langle P_\Omega u - z, u - P_\Omega u\rangle \geq 0, \ u \in H, z \in \Omega$; and for any $u \in H, z \in \Omega$ we have that $\|u - P_\Omega u\|^2 + \|P_\Omega u - z\|^2 \leq \|u - z\|^2, \ u \in H, z \in \Omega$. The set of fixed points for T from Ω into H is denoted by $\mathrm{Fix}(T)$, i.e., $\mathrm{Fix}(T) := \{u \in \Omega : Tu = u\}$.

Lemma 2.1 (see, [20]) *Let Ω be a closed convex subset of a real Hilbert space H and let $T : \Omega \to \Omega$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. If $\{u^k\}$ is a sequence in Ω weakly converging to u and if $(I-T)u^k$ converges strongly to v, then $(I-T)u = v$. In particular, if $v = 0$, then $u \in \mathrm{Fix}(T)$.*

Lemma 2.2 (see, [21]) *Let H be a real Hilbert space and $\{z^k\}$ a sequence in H such that there exists a nonempty closed set $\Omega \subseteq H$ satisfying $\omega_\omega(z^k) \subset \Omega$ and $\lim_{k\to\infty} \|z^k - z\|$ exists for every $z \in \Omega$. Then there exists $\tilde{z} \in \Omega$ such that $\{z^k\}$ converges weakly to $\tilde{z}$.*

3. A string-averaging scheme for the MSSEP

Let the *string* $J_1^t = (i_1^t, i_2^t, \cdots, i_{\gamma(J_1^t)}^t)$ be a finite nonempty subset of J_1, for every $t = 1, 2, \cdots, S_1$, where the length of the string J_1^t, denoted by $\gamma(J_1^t)$, is the number of elements in J_1^t. Put $T_1^t := P_{i_{\gamma(J_1^t)}^t} \cdots P_{i_2^t} P_{i_1^t}$, where $P_{i_l^t} = P_{C_{i_l^t}}$, for $l = 1, 2, \cdots, \gamma(J_1^t)$ and $t = 1, 2, \cdots, S_1$. Given a positive weight vector $\beta = (\beta_1, \beta_2, \cdots, \beta_{S_1})$ with $\sum_{t=1}^{S_1} \beta_t = 1$, we define the algorithmic mapping $\mathcal{P}_1 := \sum_{t=1}^{S_1} \beta_t T_1^t$. We suppose that every element of J_1 appears in at least one of the strings J_1^t. Analogously, for the family $\{Q_j\}_{j\in J_2}$, we can construct the mapping $\mathcal{P}_2 := \sum_{t=1}^{S_2} \eta_t T_2^t$ where $T_2^t := P_{j_{\gamma(J_2^t)}^t} \cdots P_{j_2^t} P_{j_1^t}$, $P_{j_l^t} = P_{Q_{j_l^t}}$ for $t = 1, 2, \cdots, S_2$, $l = 1, 2, \cdots, \gamma(J_2^t)$ and $\eta = (\eta_1, \eta_2, \cdots, \eta_{S_2})$ is also a positive weight vector such that $\sum_{t=1}^{S_2} \eta_t = 1$.

First, we need to prove the following lemma.

Lemma 3.1 *$z = [u, v] \in \Gamma$* if *and only if $(I - \mathcal{P}_1)u = (I - \mathcal{P}_2)v = 0$ and $Au = Bv$.*

Now, we consider a string-averaging scheme, $z^1 = [x^1, y^1], x^1 \in H_1, y^1 \in H_2$, and

$$\begin{aligned} x^{k+1} &= \mathcal{P}_1(x^k - \gamma_k A^*(Ax^k - By^k)), \\ y^{k+1} &= \mathcal{P}_2(y^k + \gamma_k B^*(Ax^k - By^k)), \end{aligned} \tag{3.1}$$

where γ_k is chosen by

$$\gamma_k = \frac{\rho_k f(x^k, y^k)}{a_k + \varepsilon_k} \tag{3.2}$$

with $\rho_k, f(x, y)$ and a_k defined in (1.4) and an assumption:
(ε): $\{\varepsilon_k\}$ is a bounded sequence of positive real numbers and has $\liminf_{k\to\infty} \varepsilon_k > 0$.

Theorem 3.1 *Let H_1, H_2 and H_3 be real Hilbert spaces, let A and B be two bounded linear mappings from H_1 and H_2 into H_3, respectively, and let C_i and Q_j be two closed convex subsets in H_1 and H_2, respectively, for each $i \in J_1$ and $j \in J_2$. Assume that there holds assumption (ε). Then, the sequence $\{z^k = [x^k, y^k]\}$, defined by* (3.1) *and* (3.2), *as $k \to \infty$, converges weakly to a solution of* (1.1).

4. Numerical Examples

For computation, we consider the case $H_1 = \mathbb{E}^2$, $H_2 = \mathbb{E}^3$ and $H_3 = \mathbb{E}^4$; A and B are given bellow.

$$A = \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & 0.4 \\ 0.3 & 0.6 \\ 0 & 0.1 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.1 & 0.2 \\ 0 & 0.2 & 0.4 \\ 0 & 0.1 & 0 \end{bmatrix}.$$

We consider MSSEP (1.1) with $C_i = \{x \in \mathbb{E}^2 : \langle a^i, x\rangle \leq \beta_i\}$, where $a^i = (1/i; -1)$ and $\beta_i = 0$, for $i = 1, \cdots, 10$, and $Q_j = \{y \in \mathbb{E}^3 : \|y - a^j\| \leq 1\}$, where $a^j = (1/(j+1); 1/(j+1); 1/(j+1))$ for $j = 1, \cdots, 15$. Clearly, problem (1.1) with the data above has many solutions. So, in order to verify the convergence to a solution, that we do not know, for algorithmic scheme (3.1)–(3.2), we use the errors: $error1 := \|x^{k+1} - x^k\|/\|x^k\|$ and $error2 := \|y^{k+1} - y^k\|/\|y^k\|$ with $\rho_k = 3+1/(k+1)$, $\varepsilon_k = 1$ for all $k \geq 1$, $x^1 = (-3.0; 3.0)$ and $y^1 = (-2.0; -2.5; 2.0)$. Put $\tilde{\mathcal{P}}_1 = (P_{C_5} \cdots P_{C_1} + P_{C_{10}} \cdots P_{C_6})/2$ and $\tilde{\mathcal{P}}_2 = (P_{Q_5} \cdots P_{Q_1} + P_{Q_{10}} \cdots P_{Q_6} + P_{Q_{15}} \cdots P_{Q_{11}})/3$. The numerical results with different $\mathcal{P}_1$ and $\mathcal{P}_2$ are given in the following tables.

k	$error1$	$error2$	k	$error1$	$error2$
10	0.0012953412	0.0084375860	100	0.0000584719	0.0004042637
20	0.0005700299	0.0049270390	200	0.0000189949	0.0001356754
30	0.0003496738	0.0030891459	300	0.0000100827	0.0000746127
40	0.0002398504	0.0020088602	400	0.0000064987	0.0000495669
50	0.0001747594	0.0013715507	500	0.0000046404	0.0000363808

Table 1. Method (3.1)–(3.2) with $\mathcal{P}_1 = \sum_{i=1}^{10} P_{C_i}/10$ and $\mathcal{P}_2 = \sum_{j=1}^{15} P_{Q_j}/15$

k	$error1$	$error2$	k	$error1$	$error2$
10	0.0009321189	0.0054130662	100	0.0000422591	0.0002421531
20	0.0003776241	0.0021946777	200	0.0000164338	0.0000934767
30	0.0002192796	0.0012719729	300	0.0000095113	0.0000504397
40	0.0001483827	0.000858825440	400	0.0000064435	0.0000367375
50	0.0001093893	0.000631803850	500	0.0000047357	0.0000272121

Table 2. Method (3.1)–(3.2) with $\mathcal{P}_1 = P_{C_{10}} \cdots P_{C_1}$ and $\mathcal{P}_2 = P_{Q_{15}} \cdots P_{Q_1}$

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